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LINE PARTITIONS OF INTERNAL POINTS TO A CONIC IN $PG(2,q)$

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All sets of lines providing a partition of the set of internal points to a conic C in $PG(2,q)$, q odd, are determined. There exist only three such linesets up to projectivities, namely the set of all non-tangent lines to C through an external point to C, the set of all non-tangent lines to C through a point in C, and, for square q, the set of all non-tangent lines to C belonging to a Baer subplane $PG(2,\sqrt{q})$ with $\sqrt{q}+1$ common points with C. This classification theorem is the analogous of a classical result by Segre and Korchmáros [9] characterizing the pencil of lines through an internal point to C as the unique set of lines, up to projectivities, which provides a partition of the set of all non-internal points to C. However, the proof is not analogous, since it does not rely on the famous Lemma of Tangents of Segre which was the main ingredient in [9]. The main tools in the present paper are certain partitions in conics of the set of all internal points to C, together with some recent combinatorial characterizations of blocking sets of non-secant lines, see [2], and of blocking sets of external lines, see [1].

1. Introduction

In 1977 Segre and Korchmáros gave the following combinatorial characterization of external lines to an irreducible conic in $PG(2,q)$, see [9], [4, Theorem 13.40], and [6].

Theorem 1.1. *If every secant and tangent of an irreducible conic meets a pointset $\mathcal{L}$ in exactly one point, then $\mathcal{L}$ is linear, that is, it consists of all points of an external line to the conic.*

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For even q, this was proven by Bruen and Thas [3], independently.

It is natural to ask for a similar characterization of a minimal pointset $\mathcal{L}$ meeting every *external line* to an irreducible conic C in exactly one point. In this case, we have two linear examples: a chord minus the common points with C, and a tangent minus the tangency point (and, for q even, minus the nucleus of C, as well).

For q even, it is shown in [5] that there is exactly one more possibility for $\mathcal{L}$, namely, for any even square q, the set consisting of the points of a Baer subplane π sharing $\sqrt{q}+1$ points with C, minus $\pi \cap \mathrm{C}$ and the nucleus of C.

The aim of the present paper is to prove an analogous result for q odd.

Henceforth, q is always assumed to be odd, that is, $q=p^h$ with $p>2$ prime. Then the orthogonal polarity associated to C turns $\mathcal{L}$ into a *line partition* of the set of all internal points to C. In terms of a line partition, Theorem 1.1 states that if $\mathcal{L}$ is a line partition of the set of all non-internal points to C, then $\mathcal{L}$ is a pencil of lines through an internal point to C.

Our main result is the following theorem.

Theorem 1.2. *Let $\mathcal{L}$ be a line partition of the set of internal points to a conic* C *in $PG(2,q)$, q odd. Then either*

(i) *$\#\mathcal{L}=q-1$, and $\mathcal{L}$ consists of the $q-1$ lines through an external point of* C *which are not tangent to* C*, or*
(ii) *$\#\mathcal{L}=q$, and $\mathcal{L}$ consists of the q lines through a point of* C *distinct from the tangent to* C*, or*
(iii) *$\#\mathcal{L}=q$ for a square q, and $\mathcal{L}$ consists of all non-tangent lines to* C *belonging to a Baer subplane $PG(2,\sqrt{q})$ with $\sqrt{q}+1$ common points with* C*.*

2. Prelimiaries

Throughout, C is an irreducible conic in $PG(2,q)$, q odd, and $\mathcal{L}$ is a line partition of the set of internal points to C. First the possible sizes of $\mathcal{L}$ are determined.

Lemma 2.1. *The size of $\mathcal{L}$ is either $q-1$ or q. In the latter case, $\mathcal{L}$ consists of q secant lines to* C*.*

Proof. The number of internal points to a conic is $q(q-1)/2$, see [4]. Also, a secant line of C contains $(q-1)/2$ internal points of C, whereas the number of internal points on an external line is $(q+1)/2$. No internal point belongs to a tangent to C. Let $\mathcal{L}$ consist of h secants together with k external lines to C. As $\mathcal{L}$ is a line partition of the internal points to C,

$$\frac{q(q-1)}{2} = h\frac{q-1}{2} + k\frac{q+1}{2}\,,$$

that is

$$q = h + k + \frac{2k}{q-1}\,.$$

As $\frac{2k}{q-1}$ is an integer, either $k=0$ and $h=q$, or $k=(q-1)/2=h$. ∎

The classification problem for $\#\mathcal{L}=q-1$ is solved via the characterization of blocking sets of minimal size of the external lines to a conic, as given in [1]. The dual of Theorem 1.1 in [1] reads as follows.

Proposition 2.2. *Let $\mathcal{R}$ be a lineset of size $q-1$ such that any internal point to C belongs to some line of $\mathcal{R}$. If either $q=3$ or $q>9$, then $\mathcal{R}$ consists of the $q-1$ lines through an external point of C which are not tangent to C. For $q=5,7$ there exists just one more example, up to projectivities, for which all of the lines in $\mathcal{R}$ are external to C.*

From now on, assume that $\#\mathcal{L}=q$. Note that Lemma 2.1 yields that every line of $\mathcal{L}$ is a secant line of C. The combinatorial characterization of blocking sets of non-secant lines to C, as given in [2], will be a key tool in the proof of Theorem 1.2.

Proposition 2.3. *Let $\mathcal{R}$ be a lineset of size q such that any non-external point to C belongs to some line of $\mathcal{R}$. Then one of the following occurs.*

(a) *$\mathcal{R}$ consists of q lines through a point of C distinct from the tangent to C,*
(b) *$\mathcal{R}$ consists of the non-tangent lines to C belonging to a Baer subplane $PG(2,\sqrt{q})$ with $\sqrt{q}+1$ common points with C,*
(c) *$\mathcal{R}$ consists of the $q-1$ lines through an external point P to C which are not tangent to C, together with the polar line of P with respect to C.*

Throughout, we assume that C has affine equation $Y=X^2$, and denote by Y_∞ the infinite point of C. Consider the pencil of conics $\mathcal{F}$ consisting of the conics $C_s\colon Y = X^2 - s$, with s ranging over $\mathbb{F}_q$. Some elementary properties of $\mathcal{F}$ which will be useful in the sequel are pointed out.

Lemma 2.4. *Any line of $PG(2,q)$ not passing through Y_∞ is tangent to exactly one conic of $\mathcal{F}$.*

Proof. It is enough to note that the line of equation $Y=\alpha X+\beta$ is tangent to C_s if and only if $s=-\frac{\alpha^2+4\beta}{4}$. ∎

Recall that in the finite field $\mathbb{F}_q$ half of the non-zero elements are quadratic residues or squares, and half are quadratic non-residues or non-squares. The quadratic character of $\mathbb{F}_q$ is the function χ given by

$$\chi(x)=\begin{cases} 0 \text{ if } x=0, \\ 1 \text{ if } x \text{ is a quadratic residue,} \\ -1 \text{ if } x \text{ is a quadratic non-residue.} \end{cases}$$

The following result arises from straightforward computation (see also [10, Result 1]).

Lemma 2.5. *Let* C_s *and* $\mathrm{C}_{s'}$ *be two distinct conics in* $\mathcal{F}$. *Then the affine points of* $\mathrm{C}_{s'}$ *are all either external or internal to* C_s, *according as* $\chi(s'-s)=1$ *or* $\chi(s'-s)=-1$.

As a matter of terminology, we will say that a conic C_s is *internal* (*external*) to $\mathrm{C}_{s'}$ if all the affine points of C_s are internal (external) to $\mathrm{C}_{s'}$. Let $\mathcal{I}=\{\mathrm{C}_s \mid \chi(s)=-1\}$. Clearly, the set of internal points to C consists of the affine points of the conics in $\mathcal{I}$.

3. Proof of Theorem 1.2

Throughout, we keep the notation of Section 2. In order to prove Theorem 1.2 we need to show that any line partition of size q of the internal points of C actually covers all the points of C as well. This is easily obtained for $q\equiv 1 \pmod 4$.

Lemma 3.1. *Let* $\#\mathcal{L}=q$. *If* $q\equiv 1 \pmod 4$, *then any point of* C *belongs to some line of* $\mathcal{L}$.

Proof. Assume that a point $P\in \mathrm{C}$ does not belong to any line of $\mathcal{L}$. Without loss of generality, let $P=Y_\infty$. Then the q affine points of any conic $\mathrm{C}_s\in\mathcal{I}$ partition into sets $l\cap \mathrm{C}_s$, with l ranging over $\mathcal{L}$. As q is odd, there exists a line $l_s\in\mathcal{L}$ which is tangent to C_s. Therefore C is external to C_s, but for $q\equiv 1 \pmod 4$ this implies that C_s is also external to C, contradicting $\mathrm{C}_s\in\mathcal{I}$. ∎

Assume now that $q\equiv 3 \pmod 4$. Note that this is equivalent to $\chi(-1)=-1$, see [4]. Then Lemma 2.5 yields that C_s is internal to $\mathrm{C}_{s'}$ if and only if $\mathrm{C}_{s'}$ is external to C_s.

For the proof of the following two lemmas it is convenient to recall the notion of a tournament. A tournament is a directed graph such that each pair of distinct vertices is joined by exactly one arc. A tournament that is naturally linked with the conic partition $\mathcal{I}$ of the internal points to C is the Paley tournament with q vertices, which is defined as the directed graph whose vertices are the elements in $\mathbb{F}_q$, and there is an edge from a to b if and only if $\chi(a-b)=1$.

Lemma 3.2. *Let* C_s *be a conic in* $\mathcal{I}$. *If* $q\equiv 3 \pmod 4$, *then there are exactly* $\frac{q-3}{4}$ *conics in* $\mathcal{I}$ *that are internal to* C_s.

Proof. It is well-known that the Paley tournament with q vertices is doubly regular, see e.g. [8]. In particular, for any vertex $s\in\mathbb{F}_q$ the number of vertices dominated by both s and 0 is equal to $\frac{q-3}{4}$. Then the assertion follows from Lemma 2.5. ∎

The following lemma will be crucial in the proof of Theorem 1.2.

Lemma 3.3. *Let* $S=\left\{s_1,s_2,\dots,s_{\frac{q-1}{2}}\right\}$ *be the set of non-squares in* $\mathbb{F}_q$. *If* $q\equiv 3 \pmod 4$, *then any real function* φ *on* S *such that*

$$\sum_{\chi(s_i-s_j)=-1}\varphi(s_i)=\sum_{\chi(s_i-s_j)=1}\varphi(s_i),\qquad \text{for any } j=1,\dots,\frac{q-1}{2} \tag{1}$$

is constant.

Proof. Let $A=(a_{ij})$ be the $\frac{q-1}{2}\times\frac{q-1}{2}$ matrix given by $a_{ij}=\chi(s_i-s_j)$. Note that (1) holds if and only if the vector

$$\left(\varphi(s_1),\dots,\varphi(s_i),\dots,\varphi\left(s_{\frac{q-1}{2}}\right)\right)$$

belongs to the null space of A. By Lemmas 2.5 and 3.2 such a condition is fulfilled when φ is constant. Then to prove the assertion it is enough to show that the real rank of A is at least $\frac{q-1}{2}-1$. Note that A is the incidence matrix of a sub-tournament of the Paley tournament. As the incidence matrix of any tournament with n vertices is either $n-1$ or n (see [7]), the claim follows. ∎

Lemma 3.4. *Let* $\#\mathcal{L}=q$. *If* $q\equiv 3 \pmod 4$, *then the number of lines of* $\mathcal{L}$ *through any point* P *of* C *is* 1, $\frac{q+1}{2}$ *or* q.

Proof. Assume without loss of generality that $P=Y_\infty$. Let $\mathcal{L}_P$ be the set of lines of $\mathcal{L}$ passing through P, and set $m=\#\mathcal{L}_P$. For any $l\in\mathcal{L}\setminus\mathcal{L}_P$, denote by $\mathrm{C}^{(l)}$ the conic of $\mathcal{F}$ which is tangent to l according to Lemma 2.4. Also, for any s non-square in $\mathbb{F}_q$ denote by $\mathcal{I}_s$ the set of conics of $\mathcal{I}$ which are internal to C_s.

As l is a tangent to $\mathrm{C}^{(l)}$ meeting C at some point, the conic C is external to $\mathrm{C}^{(l)}$. Then $\mathrm{C}^{(l)}$ belongs to $\mathcal{I}$ since $q \equiv 3 \pmod 4$. We claim that for any $\mathrm{C}_s \in \mathcal{I}$ and for any $l \in \mathcal{L} \setminus \mathcal{L}_P$, l not tangent to C_s,

$$\mathrm{C}_s \text{ is external to } \mathrm{C}^{(l)} \text{ if and only if } l \text{ is a secant of } \mathrm{C}_s\,. \tag{2}$$

Clearly, if l is a secant of C_s, then both the points of $\mathrm{C}_s \cap l$ are external to $\mathrm{C}^{(l)}$. Therefore C_s is external to $\mathrm{C}^{(l)}$. To prove the only if part of (2), note that for any $l \in \mathcal{L} \setminus \mathcal{L}_P$ the set of $\frac{q-1}{2}$ points of l which are internal to C consists of one point lying on $\mathrm{C}^{(l)}$ together with $\frac{q-3}{4}$ point pairs, each of which contained in a conic of $\mathcal{I}$. Taking into account Lemma 3.2, this means that l is a secant of all the conics of $\mathcal{I}$ that are external to $\mathrm{C}^{(l)}$.

Now, for any $\mathrm{C}_s \in \mathcal{I}$ let $\psi(\mathrm{C}_s)$ be the number of lines of $\mathcal{L}$ which are tangent to C_s. Then,

$$\sum_{\mathrm{C}_{s'} \in \mathcal{I}_s} \psi(\mathrm{C}_{s'}) = \sum_{\mathrm{C}_{s'} \in \mathcal{I} \setminus \mathcal{I}_s, \mathrm{C}_{s'} \neq \mathrm{C}_s} \psi(\mathrm{C}_{s'}), \qquad \text{for any } \ \mathrm{C}_s \in \mathcal{I}. \tag{3}$$

In fact, (2) yields that $\sum_{\mathrm{C}_{s'} \in \mathcal{I}_s} \psi(\mathrm{C}_{s'})$ equals the number of lines in $\mathcal{L} \setminus \mathcal{L}_P$ which are secants to C_s, that is $\frac{q-m-\psi(\mathrm{C}_s)}{2}$. As the total number of lines in $\mathcal{L}$ which are tangent to a conic of $\mathcal{I}$ distinct from C_s is $q-m-\psi(\mathrm{C}_s)$, Equation (3) follows. Taking into account Lemma 2.5, Lemma 3.3 implies that $\psi(\mathrm{C}_s) \equiv t$ for a constant t. By Lemma 2.4,

$$\sum_{\mathrm{C}_s \in \mathcal{I}} \psi(\mathrm{C}_s) = t\frac{q-1}{2} = q - m\,,$$

which implies that either (a) $t=2$, $m=1$, (b) $t=0$, $m=q$, or (c) $t=1$, $m=\frac{q+1}{2}$. ∎

Proposition 3.5. *Let $\#\mathcal{L}=q$. Then $\mathcal{L}$ consists either of the q lines through a point of* C *distinct from the tangent to* C*, or of the non-tangent lines to C belonging to a Baer subplane $PG(2,\sqrt{q})$ with $\sqrt{q}+1$ common points with* C.

Proof. Lemmas 3.1 and 3.4 yields that $\mathcal{L}$ satisfies the hypothesis of Proposition 2.3. Actually, (c) of Proposition 2.3 cannot occur as in this case not every line of $\mathcal{R}$ is a secant line to C. Hence the assertion is proved. ∎

We are now in a position to prove Theorem 1.2. Note first that each lineset $\mathcal{L}$ as in (i), (ii) and (iii) gives rise to a line partition of the internal points to C; when $\mathcal{L}$ is as in (iii) this depends on the fact that no point in $PG(2,\sqrt{q})$ is internal to C. Theorem 1.2 then follows from Lemma 2.1, together with Propositions 2.2 and 3.5.

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